

STRONG DIAMETER TWO PROPERTY AND CONVEX COMBINATION OF SLICES REACHING THE UNIT SPHERE

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ABSTRACT. We characterise the class of those Banach spaces in which every convex combination of slices of the unit ball intersects the unit sphere as the class of those ones in which all the convex combinations of slices of the unit ball have diameter two and the diameter is attained.

1. INTRODUCTION

It is a well known result in geometry of Banach spaces that every non-empty relatively weakly open subset of the unit ball contains a convex combination of slices of the unit ball (c.f., e.g. [4, Lemma II.1]). Although the reverse inclusion does not hold in general, it may even happen for some Banach spaces that every convex combination of slices of the unit ball is relatively weakly open. The main result of [1] shows that this is the case for $C(K)$ for every scattered compact topological space K . In [1, Section 3] the following properties are introduced:

- (1) Every convex combination of slices of the unit ball is weakly open.
- (2) Every convex combination of slices of the unit ball contains some weak neighbourhood.
- (3) Every convex combination of slices of the unit ball intersects the unit sphere.

Notice that (1) implies (2) which in turn implies (3). Moreover (see [1, Section 3]) the authors wonder which class of spaces enjoy the above properties and if such spaces have any relation with the diameter two properties.

The aim of this note is to show that the diameter two properties have a strong connection with the above properties and to characterise the property (3) in terms of a "diameter two property" kind condition. Indeed, we show in Theorem 2.1 that a Banach space X has the strong diameter two property (i.e. every convex combination of slices of the unit ball has diameter two) if, and only if, every convex combination of slices of the unit ball C contains

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points arbitrarily close to the unit sphere of the space. The ideas involving the proof allow us to prove in Theorem 2.4 that a Banach space X enjoys the property (3) if, and only if, every convex combination of slices of the unit ball has diameter two and the diameter is attained. We also give an example of a Banach space with the strong diameter two property but failing the above property in Corollary 2.3. Finally, following similar ideas to the ones of the case of the strong diameter two property, we show that the property (3) is preserved by taking projective tensor product from both factors but not from only one of them.

Notation: We will consider only real Banach spaces. Given a Banach space X , we will denote the closed unit ball (respectively the unit sphere) by B_X (respectively S_X). We will also denote by X^* the topological dual of X . Given two Banach spaces X and Y we will denote by $L(X, Y)$ the space of all linear and continuous operators from X to Y , and by $X \hat{\otimes}_\pi Y$ the projective tensor product of X and Y (see [7] for a detailed treatment of the previous spaces). Given a bounded subset C of X , we will mean by a slice of C a set of the following form

$$S(C, f, \alpha) := \{x \in C : f(x) > 1 - \alpha\}$$

where $f \in X^*$ and $\alpha > 0$. When C is a bounded and convex subset of X , by a convex combination of slices of C we will mean a set of the following form

$$\sum_{i=1}^n \lambda_i S_i,$$

where $\lambda_1, \dots, \lambda_n \in [0, 1]$ satisfy that $\sum_{i=1}^n \lambda_i = 1$ and each S_i is a slice of C . A Banach space X has the *strong diameter two property* if every convex combination of slices of the unit ball has diameter two. We refer the reader to [2, 3] and references therein for background about diameter two properties.

2. MAIN RESULTS

In [1, Section 3] it is stated to be unclear whether there is any connection between having weakly open convex combination of slices and the diameter two properties. The following easy argument shows that the strong diameter two property is a necessary condition.

Theorem 2.1. *Let X be a Banach space. The following assertions are equivalent:*

- (1) *X has the strong diameter two property.*
- (2) *For every convex combination of slices C of B_X and every $\varepsilon > 0$ there exists $x \in C$ such that $\|x\| > 1 - \varepsilon$.*

Proof. (1) \Rightarrow (2) is obvious, so let us prove (2) \Rightarrow (1). To this aim pick $C := \sum_{i=1}^n S(B_X, f_i, \alpha)$ to be a convex combination of slices of B_X and $\varepsilon > 0$. Define $D := \frac{1}{2} (\sum_{i=1}^n \lambda_i S(B_X, f_i, \alpha) + \sum_{i=1}^n \lambda_i S(B_X, -f_i, \alpha))$, which is also a

convex combination of slices of B_X . Choose $x = \frac{1}{2}(\sum_{i=1}^n \lambda_i x_i + \sum_{i=1}^n \lambda_i y_i) \in D$ such that $\|x\| > 1 - \varepsilon$. Notice that, by the definition of D , we get that $-\sum_{i=1}^n \lambda_i y_i, \sum_{i=1}^n \lambda_i x_i \in C$. Consequently

$$\text{diam}(C) \geq \left\| \sum_{i=1}^n \lambda_i x_i - \left(-\sum_{i=1}^n \lambda_i y_i \right) \right\| = 2\|x\| > 2(1 - \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary we conclude the desired result. ■

Proposition 2.1 shows that if a Banach space X satisfies that every convex combination of slices intersects the unit sphere then X has the SD2P. The converse, however, is not longer true. For this we need the following lemma.

Lemma 2.2. *Let X be a strictly convex Banach space. Then there are convex combinations of slices in B_X without points in S_X .*

Proof. Consider two disjoint slices S_1, S_2 of B_X and $C := \frac{S_1 + S_2}{2}$, and we claim that $C \cap S_X = \emptyset$. Indeed, if there existed $z \in C \cap S_X$ then there would exist $x \in S_1, y \in S_2$ such that $z = \frac{x+y}{2}$. Since $z \in S_X$ is an extreme point then $x = y = z$, which is impossible because S_1 and S_2 were taken to be disjoint. ■

Corollary 2.3. *There exist Banach spaces X with the SD2P containing convex combinations of slices in B_X without points in S_X .*

Proof. In [5, P. 168] an example of a strictly convex space being a non-reflexive M-embedded Banach space (and hence with the SD2P) X is exhibited. From the above lemma, this Banach space satisfies the required conditions. ■

In [1, Question (iii)] it is asked which Banach spaces verify that every convex combination of slices of its unit ball has a non-empty. A slight modification of the previous easy proof yields a characterisation of those spaces in terms of the diameter of convex combination of slices.

Theorem 2.4. *Let X be a Banach space. The following are equivalent:*

- (1) *Every convex combination of slices of B_X intersects to S_X .*
- (2) *For every convex combination of slices C of B_X there are $x, y \in X$ such that $\|x - y\| = 2$.*

Proof. (2) implies (1) is clear. For (1) implies (2), consider a convex combination of slices of B_X given by $C := \sum_{i=1}^n S(B_X, f_i, \alpha)$. Define $D := \frac{1}{2}(\sum_{i=1}^n \lambda_i S(B_X, f_i, \alpha) + \sum_{i=1}^n \lambda_i S(B_X, -f_i, \alpha))$, which is also a convex combination of slices of B_X . Choose, from the assumption,

$$x_0 = \frac{1}{2} \left(\sum_{i=1}^n \lambda_i x_i + \sum_{i=1}^n \lambda_i y_i \right) \in D \cap S_X.$$

Now $x := \sum_{i=1}^n \lambda_i x_i \in C$, $y := -\sum_{i=1}^n \lambda_i y_i \in C$ and $\|\frac{x - (-y)}{2}\| = 1$. ■

Let us conclude with some consequences related to preservance of this property by projective tensor products. The next proposition follows similar ideas to the ones of [2, Theorem 3.5].

Proposition 2.5. *Let X and Y be two Banach spaces with the property that every convex combination of slices of the unit ball intersects the unit sphere. Then every convex combination of slices of $B_{X\widehat{\otimes}_\pi Y}$ intersects the unit sphere.*

Proof. Consider $C := \sum_{i=1}^n S(B_{X\widehat{\otimes}_\pi Y}, T_i, \alpha)$ to be a convex combination of slices of $B_{X\widehat{\otimes}_\pi Y}$ and let us prove that $C \cap S_{X\widehat{\otimes}_\pi Y} \neq \emptyset$. To this aim consider $u_i \otimes v_i \in S(B_{X\widehat{\otimes}_\pi Y}, T_i, \alpha) \cap (S_X \otimes S_Y)$ for all $i \in \{1, \dots, n\}$. Now

$$u_i \otimes v_i \in S(B_{X\widehat{\otimes}_\pi Y}, T_i, \alpha) \Leftrightarrow T_i(u_i)(v_i) > 1 - \alpha \Leftrightarrow u_i \in S(B_X, v_i \circ T_i, \alpha).$$

By assumptions there exists an element $\sum_{i=1}^n \lambda_i x_i \in \sum_{i=1}^n \lambda_i S(B_X, v_i \circ T_i, \alpha)$ whose norm is 1. By Hahn-Banach theorem we can find a functional $x^* \in S_{X^*}$ such that $x^*(x_i) = 1$ holds for all $i \in \{1, \dots, n\}$. It is obvious that $\sum_{i=1}^n \lambda_i x_i \otimes v_i \in C$. Now, by assumptions and by the same procedure we get elements $y_1, \dots, y_n \in B_Y$ and a functional $y^* \in S_{Y^*}$ such that $y^*(y_i) = 1$ holds for every $i \in \{1, \dots, n\}$ and such that $\sum_{i=1}^n \lambda_i x_i \otimes y_i \in C$. Now

$$\left\| \sum_{i=1}^n \lambda_i x_i \otimes y_i \right\| \geq \sum_{i=1}^n \lambda_i x^*(x_i) y^*(y_i) = 1.$$

Consequently $C \cap S_{X\widehat{\otimes}_\pi Y} \neq \emptyset$ as desired. ■

Remark 2.6. The assumption on both factors is necessary. In fact, consider $X = \ell_\infty$ and $Y = \ell_p^3$ for some $2 < p < \infty$. Note that every convex combination of slices of B_X intersects the unit sphere [1, Example 3.3]. However, this is not longer true for $X\widehat{\otimes}_\pi Y$ because such space even fails the strong diameter two property [6, Corollary 3.9], so Theorem 2.1 yields the existence of a convex combination of slices C in $B_{X\widehat{\otimes}_\pi Y}$ and a radius $0 < r < 1$ such that $C \subseteq rB_{X\widehat{\otimes}_\pi Y}$.

REFERENCES

- [1] T. A. Abrahamsen and V. Lima, *Relatively weakly open convex combination of slices*, preprint.
- [2] J. Bercerra Guerrero, G. López-Pérez and A. Rueda Zoca, *Octahedral norms in spaces of operators*, J. Math. Anal. Appl. **427** (2015), 171-184.
- [3] J. Bercerra Guerrero, G. López-Pérez and A. Rueda Zoca, *Subspaces of Banach spaces with big slices*, Banach J. Math. Anal. **10** (2016), n 4, 771-782.
- [4] N. Ghoussoub, G. Godefroy, B. Maurey, W. Schachermayer, *Some topological and geometrical structures in Banach spaces*, Mem. Amer. Math. Soc. **378** (1987).
- [5] P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Math. **1547**, Springer-Verlag, Berlin-Heidelberg, 1993.
- [6] J. Langemets, V. Lima and A. Rueda Zoca, *Octahedral norms in tensor products of Banach spaces*, preprint.

- [7] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 2002.

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